TRANSIENT MINOR DISTURBANCES IN FLAT STREAMS OF A HIGHLY CONDUCTIVE PLASMA IN A CHANNEL

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Transient magnetohydrodynamic disturbances of flat streams of a highly conductive, nonviscous, thermally nonconductive, quasineutral plasma in a channel of slowly varying cross section with sectioned electrodes are analyzed in a linear approximation. The influence of the Hall effect is taken into account in the analysis. It is shown that the evolution of a disturbance in the isomagnetic parameter B/ρ is comprised of transport along the channel together with the plasma stream, transport along the undisturbed electron trajectories, and diffusion due to the finite conductance of the plasma. The time of establishment of the flow is equal to the time of flight of the plasma through the channel (the region occupied by the magnetic field). The present report is a generalization of the analysis of steady disturbances conducted in [1].

1. A considerable number of works have been devoted to transient flows of a plasma. This is explained by the necessity of the analysis of processes of plasma acceleration in pulsed systems, the clarification of the possibility of establishing flows in steady-state accelerators, and the analysis of problems of the stability of steady flows. We shall consider low-frequency transient processes which do not disturb the quasineutrality of the plasma and are subject to a hydrodynamic description.

The analysis of transient flows has been performed by different authors without allowance for the influence of the Hall effect on the flow. The transient one-dimensional acceleration of a plasma with a constant conductance was analyzed in [2]. It was shown that allowance for the three-dimensional distribution of the electric current leads to a flow which differs strongly from that calculated on the basis of the model of a current layer. With slow variation in the characteristics of the discharge the effect of the initial conditions on the flow is important for times shorter than the time of flight of the plasma through the channel.

The authors of [3], in which a numerical calculation was made of the two-dimensional flows of a plasma with a constant conductance without allowance for the Hall effect, came to an analogous conclusion concerning the establishment of flows in a time on the order of the time of flight of the plasma through the channel. If the transfer coefficients of the plasma depend on the temperature, then the situation can change: short-wavelength hydrodynamic oscillations in a plasma stream can prove to be unstable.

A nonlinear numerical calculation of the one-dimensional acceleration of a plasma performed in [4] shows that if the conductance of the plasma increases with an increase in temperature, then the initial conditions have a considerable effect on the nature of the flow. If the current is initially distributed in a narrow layer, then a self-sustaining current T-layer with high conductance and temperature develops as a result of the heating of the plasma by the current. Two shock waves propagate from the site of formation of the T-layer; the wave moving toward the channel entrance can, by heating the plasma, cause the formation of a second T-layer, and so forth. As a result the discharge current is concentrated in several T-layers, and the accelerating plasma is distributed along the channel in the form of clusters following one after another.

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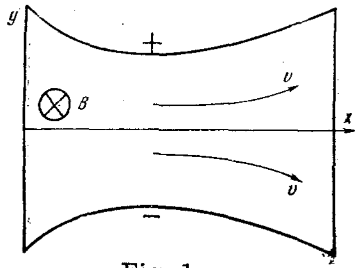

Fig. 1

Making allowance for the Hall effect can also lead to the loss of stability of plasma flows. A numerical calculation of two-dimensional flows with allowance for the Hall effect [5] showed that during flow in a channel with continuous metal walls – the electrodes – stability is lost when the exchange parameter [6] exceeds a critical value which depends on the ratio of the gaskinetic and magnetic pressures and on the magnetic Reynolds number. A theoretical analysis of the stability of short-wavelength oscillations in Hall flows [7] predicts the instability of flows of a perfectly conductive plasma if there exist regions in which the vectors of the density and the total pressure (gaskinetic and magnetic) of the plasma are not parallel.

The purpose of the present report is the generalization to the case of transient disturbances of the analysis of steady weakly disturbed Hall flows of a plasma in a channel with sectioned electrodes which was performed earlier in [1]. The disturbances considered in [1] were due to the slight imperfection in the cutting of the electrodes and considerably altered the flow pattern in the presence of a strongly expressed Hall effect. It is interesting to examine the case in which, in addition to the disturbances caused by the imperfection in the cutting, there exist disturbances caused by transient irregularities at the entrance to the channel. This is possible, for example, when there are slight deviations in the mode of supply of the working substance.

Let us consider a flat plasma flow in an infinitely long channel in the presence of an intrinsic (produced by the discharge current) transverse magnetic field **B** (Fig. 1). The vector **B** is oriented along the z axis, while the vectors of the plasma velocity **v**, the electric field strength **E**, and the electric current density **j** are located in the xy plane. All the parameters of the flow depend on the coordinates x and y and the time t; the width of the channel in the direction of the z axis is considered as infinite. We will analyze the flow of a fully ionized, quasineutral, nonviscous, and thermally nonconductive plasma. We will assume the conductance σ of the plasma to be constant, and we will neglect the inertia of the electrons. With these assumptions the flow is described by the following system of equations:

$$\begin{gathered}\rho\left(\frac{\partial}{\partial t}+v\nabla\right)v=-\nabla P,\quad \frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0\\ \frac{\mathbf{j}}{\sigma}=\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}+\frac{M}{e\rho}(\nabla P-\nabla p_i)\\ P=p_i+p_e+\frac{B^2}{8\pi},\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \operatorname{rot}\mathbf{B}=\frac{4\pi}{c}\mathbf{j}\\ \operatorname{div}\mathbf{B}=0,\qquad p_i=p_i(\rho),\quad p_e=p_e(\rho).\end{gathered}\tag{1.1}$$

Here ρ is the plasma density, and $p_{i,e}$ are the gaskinetic partial pressures of the ion and electron components of the plasma, which we assume to be polytropically dependent on the density ρ.

Let us consider a channel of slowly varying cross section, in which the following conditions are satisfied:

$$|v_y|\ll|v_x|,\quad \left|\frac{d}{dt}v_y\right|\ll\left|\frac{d}{dt}v_x\right|.\tag{1.2}$$

We will assume that the magnetic Reynolds number Re_m is large,

$$\mathrm{Re}_m=uL/\nu_m\gg1\quad(\nu_m=c^2/4\pi\sigma),\tag{1.3}$$

where u is the characteristic longitudinal velocity of the plasma, L is the characteristic longitudinal scale of the length in which the flow parameters vary significantly, and ν_m is the magnetic viscosity of the plasma.

Let us assume that the radius of curvature r of the plasma streamlines $y_0(t, x)$, which are determined by the equation

$$\partial y_0/\partial x=v_y/v_x,\tag{1.4}$$

is large compared with the ionic Larmor radius Λ,

$$r\gg\Lambda\quad(\Lambda=Mcv_x/eB).\tag{1.5}$$

When the conditions (1.2), (1.3), and (1.5) are satisfied one can neglect the transverse component (along the y axis) in the first equation of (1.1), as well as the terms j_y/σ and $(M/e\rho)\,(\partial P/\partial y)$, in comparison with the Lorentzian term $v_x B/c$ in the y-component of the third equation of (1.1). Consequently, in this approximation we obtain the following system of equations in place of (1.1):

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)v_x = -\frac{\partial P(t,x)}{\partial x}, \quad \frac{\partial \rho}{dt} + \operatorname{div}\rho\mathbf{v} = 0 \tag{1.6}$$
$$\frac{\nu_m}{c}\frac{\partial B}{\partial y} = E_x + \frac{v_y B}{c} + \frac{M}{e\rho}\left(\frac{\partial P}{\partial x} - \frac{\partial p_i(\rho)}{\partial x}\right)$$
$$0 = E_y - \frac{v_x B}{c} - \frac{M}{e\rho}\frac{\partial p_i(\rho)}{\partial y}$$
$$P(t,x) = p_i(\rho) + \rho_e(\rho) + \frac{B^2}{8\pi},$$
$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = -\frac{1}{c}\frac{\partial B}{\partial t}.$$

Differentiating the third equation of (1.6) with respect to y and using the second, fourth, and sixth equations of (1.6), we find

$$\frac{\nu_m}{c}\frac{\partial^2 B}{\partial y^2} = \frac{\rho}{c}\left(\frac{\partial}{\partial t} - \mathbf{v}\nabla\right)\frac{B}{\rho} - \frac{M}{e\rho^2}\frac{\partial P}{\partial x}\frac{\partial \rho}{\partial y}. \tag{1.7}$$

For a perfectly conductive plasma without allowance for the Hall effect it follows from (1.7) that

$$\frac{d}{dt}\frac{B}{\rho} = 0,$$

i.e., the fact that the magnetic field is "frozen" into the plasma.

Let us consider small transient disturbances in the main steady quasi-one-dimensional flow of a plasma in a channel with sectioned electrodes. We will take the width of the sections as infinitely small. The parameters of the undisturbed flow (denoted by the subscript 0) are described by the equations

$$P_0(x) = p_0(\rho) + \frac{B_0^2}{8\pi}, \quad \varphi_0 = \varphi_{00}(x) - \frac{k_0}{c}\psi m^{\cdot} - \frac{M}{e}\int\frac{dp_i(\rho_0)}{\rho_0} \tag{1.8}$$
$$\frac{B_0}{\rho_0} = k_0 = \text{const}, \quad \frac{v_0^2}{2} + \frac{e}{M}\varphi_{00}(x) = \text{const}, \quad \frac{d\varphi_{00}}{dx} = \frac{M}{e\rho_0}\frac{dP_0}{dx}$$
$$E_0 = -\nabla\varphi_0, \quad B_0 = B_0(x), \quad \rho_0 = \rho_0(x), \; v_0 = v_0(x)$$
$$\rho_0 v_0 f = m^{\cdot} = \text{const}.$$

Here v_0 is the x-component of the velocity $\mathbf{v}_0$, f(x) is the width of the channel, and $m^{\cdot}$ is the mass flow rate per second of the working substance (the plasma). The normalized stream function ψ is determined by the equation

$$\psi = \rho_0 v_0 [y - y_k(x)]/m^{\cdot}, \tag{1.9}$$

where $y_k(x)$ is the profile of the channel cathode, so that $\psi = 0$ at the cathode and $\psi = 1$ at the anode.

Let us change from the variables t, x, y to the variables t, x, ψ. By linearizing the first and fifth equations of (1.6) we find

$$\rho_0\left(\frac{\partial v_{1x}}{\partial t} + \frac{\partial}{\partial x}v_0 v_{1x}\right) + \rho_1 v_0\frac{\partial v_0}{\partial x} = -\frac{\partial P_1}{\partial x} \tag{1.10}$$
$$P_1(t,x) = c_T^2\rho_1 + B_0 B_1/4\pi \quad (c_T^2 = dp_0/d\rho_0). \tag{1.11}$$

From the second equation of (1.6) it follows that

$$\partial\rho_1/\partial t + \operatorname{div}(\rho_1\mathbf{v}_0 + \rho_0\mathbf{v}_1) = 0. \tag{1.12}$$

A disturbance in the isomagnetic parameter $k = B/\rho$ will be defined as k_1. Then we have

$$k_1 = \frac{B_1}{\rho_0} - k_0\frac{\rho_1}{\rho_0}. \tag{1.13}$$

From (1.11) and (1.13) we obtain

$$\rho_1/\rho_0 = P_1/\rho_0 c_H^2 - k_1/k_0\,(c_A/c_H)^2 \tag{1.14}$$
$$B_1/B_0 = P_1/\rho_0 c_H^2 + k_1/k_0\,(c_T/c_H)^2$$
$$c_H^2 = c_T^2 + c_A^2, \quad c_A^2 = B_0^2/4\pi\rho_0.$$

By linearizing Eq. (1.7) and using (1.14) we can find an equation describing the evolution of the disturbance in the isomagnetic parameter:

$$\frac{\partial^2 k_1}{\partial\psi^2} = \frac{1}{\Omega(\eta)}\frac{\partial k_1}{\partial t} + \frac{\partial k_1}{\partial \eta} + a(\eta)\frac{\partial k_1}{\partial \psi} \tag{1.15}$$

$$\Omega = \nu_m \frac{\rho_0 v_0}{m^\cdot}\left(\frac{c_T}{c_H}\right)^2, \quad \frac{d\eta}{dx} = \frac{\Omega}{v_0} \tag{1.16}$$

$$a(\eta) = \frac{Mc}{em^\cdot B_0}\left(\frac{c_A}{c_H}\right)^2 \frac{dP_0}{d\eta} \geqslant 0.$$

Equation (1.15) is a generalization to the transient case of the steady-state equation obtained in [1].

2. Let us analyze the evolution of small disturbances in the isomagnetic parameter. Changing from the variables t, η, ψ to the variables θ, η, ψ, where θ is determined by the equation

$$\theta = t - \int_{x_{00}}^{x} \frac{d\zeta}{v_0(\zeta)} = t - \int_{\eta_{00}}^{\eta} \frac{d\mu}{\Omega(\mu)} \tag{2.1}$$

(we assume that at the channel entrance, i.e., at $x = x_{00}$, the velocity v_0 is different from zero, so that θ is finite everywhere), we obtain

$$\frac{\partial^2 k_1}{d\psi^2} = \frac{\partial k_1}{\partial \eta} + a(\eta)\frac{\partial k_1}{\partial \psi}. \tag{2.2}$$

The variable θ enters into Eq. (2.2) as a parameter. From this and from the definition (2.1) it follows that a transient disturbance k_1 is carried along the channel by the plasma stream. If the electron stream function ψ_e is introduced by the equations

$$\mathbf{v}_{e0} = \mathbf{v}_0 - M\mathbf{j}_0/e\rho_0, \quad \rho_0\mathbf{v}_{e0} = m^\cdot\nabla\psi_e \times \mathbf{n}_z \tag{2.3}$$

($\mathbf{n}_z$ is the unit vector in the direction of the z axis), so that

$$B_0 = 4\pi em^\cdot(\psi - \psi_e)/Mc,$$

then one can ascertain that the operator $\partial/\partial\eta + a\partial/\partial\psi$ on the right side of (2.2) corresponds to the differentiation operator along an undisturbed electron trajectory $\psi_e = \text{const}$. With a perfectly conductive plasma ($\sigma \to \infty$, $\nu_m \to 0$) we obtain

$$k_1 = k_1(\theta, \psi_e). \tag{2.4}$$

The left side of Eq. (2.2) describes the diffusion due to the finite conductance of the plasma. The evolution of the disturbance k_1 is comprised of transport together with the plasma stream, transport along the undisturbed electron trajectories, and diffusion due to the finite conductance of the plasma.

Equation (2.2) with $a = a_0 = \text{const}$ was analyzed in [1], where it was shown that in the case of weak influence of the Hall effect ($a \to 0$) the disturbances penetrate from the electrodes into the stream like a skin (the thickness of the skin layers is determined by the diffusion of the plasma in the magnetic field); in the case of a strongly expressed Hall effect ($a \to \infty$) the function k_1 has the form

$$k_1 \approx g_1(\theta, \eta - \psi/a_0) + g_2(\theta, \eta + \psi/a_0)\exp(a_0\psi). \tag{2.5}$$

The dependence (2.5) signifies that the disturbances are transported along the electron trajectories from the cathode to the anode and an electromagnetic layer is formed near the anode.

3. Let us examine the integral equations. By integrating (1.10) with respect to ψ from 0 to 1 we obtain the first integral equation

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)v_0^2 \int_0^1 \frac{v_{1x}}{v_0}\,d\psi = -v_0\left(\frac{1}{\rho_0}\frac{\partial P_1}{\partial x} + v_0\frac{dv_0}{dx}\int_0^1 \frac{\rho_1}{\rho_0}\,d\psi\right). \tag{3.1}$$

From the continuity equation (1.12) we find

$$\frac{\partial v_{1y}}{\partial\psi} = -\frac{m^\cdot}{\rho_0 v_0}\left[\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)\frac{\rho_1}{\rho_0} + \frac{\partial v_{1x}}{\partial x} - \frac{\rho_0 v_{0y}}{m^\cdot}\frac{\partial v_{1x}}{\partial\psi} + \frac{v_{1x}}{\rho_0}\frac{d\rho_0}{dx}\right]. \tag{3.2}$$

At the channel walls $y = y_-(x)$ ($\psi = 0$) and $y = y_+(x)$ ($\psi = 1$) the following conditions are satisfied:

$$v_y|_{\psi=0} = v_x|_{\psi=0} dy_-/dx,$$
$$v_y|_{\psi=1} = v_x|_{\psi=1} dy_+/dx. \tag{3.3}$$

By integrating (3.2) with respect to ψ from 0 to 1, using (3.3) and the equality

$$\frac{\partial}{\partial\psi}\frac{\rho_0 v_{0y}}{m} = -\frac{1}{\rho_0 v_0}\frac{d\rho_0 v_0}{dx},$$

we obtain the second integral equation

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)\int_0^1 \frac{\rho_1}{\rho_0}\,d\psi + v_0\frac{\partial}{\partial x}\int_0^1 \frac{v_{1x}}{v_0}\,d\psi = 0. \tag{3.4}$$

Substituting the value ρ_1/ρ_0 into (3.1) and (3.4), from (1.14) we have

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)v_0^2\int_0^1 \frac{v_{1x}}{v_0}\,d\psi = -v_0\left[\frac{1}{\rho_0}\frac{\partial P_0}{\partial x} + \frac{P_1}{\rho_0 c_H^2}v_0\frac{dv_0}{dx} - \left(\frac{c_A}{c_H}\right)^2 v_0\frac{dv_0}{dx}\int_0^1 \frac{k_1}{k_0}\,d\psi\right], \tag{3.5}$$

$$v_0\frac{\partial}{\partial x}\int_0^1 \frac{v_{1x}}{v_0}\,d\psi = -\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right)\left[\frac{P_1}{\rho_0 c_H^2} - \left(\frac{c_A}{c_H}\right)^2\int_0^1 \frac{k_1}{k_0}\,d\psi\right].$$

Equations (3.5) make it possible to obtain an equation connecting a disturbance P_1 and the value $\int_0^1 k_1 d\psi$. If the latter is known then $P_1(t, x)$ is determined from the equation obtained.

4. Let us consider further the exact solution and the ultra-Hall mode. The steady disturbances analyzed in [1] satisfy the equations

$$\int_0^1\left(\frac{\rho_1}{\rho_0} + \frac{v_{1x}}{v_0}\right)d\psi = 0 \tag{4.1}$$

$$v_0^2\frac{\partial}{\partial x}\int \frac{v_{1x}}{v_0}\,d\psi + v_0\frac{dv_0}{\partial x}\int_0^1 \frac{v_{1x}}{v_0}\,d\psi + \frac{1}{\rho_0}\frac{\partial P_1}{\partial x} = 0.$$

We will assume that the sections are weakly short-circuited through purely ohmic resistances and that the current through each respective pair of sections (anode–cathode) is constant and does not vary with time. In [1] it is shown that in this case a disturbance B_1 in the magnetic field at the electrodes is stationary. Since the transient problem is separated from the steady-state problem, one can, without, disturbing the generality of the analysis, confine oneself to the consideration of purely transient disturbances; in this case the disturbance B_1 is reduced to zero at the electrodes. The conditions on the function $k_1(t, \eta, \psi)$ have the form

$$k_1(t, \eta_{00}, \psi) = g(t, \psi), \quad k_1(0, \eta, \psi) = k^{(0)}(\eta, \psi)$$
$$k_1(t, \eta, 0) = k_1(t, \eta, 1) = -k_0 P_1(t, \eta)/\rho_0 c_T^2. \tag{4.2}$$

Let us discuss the case when $k^{(0)}(\eta, \psi) \equiv 0$. In this case $g(0, \psi) = 0$ and $P_1(0, \eta) = 0$. As the undisturbed flow we choose the case of exponential flow $a = a_0 = \text{const}$ analyzed in [1]. Its parameters have the form

$$v_0 = v_m \operatorname{th}(x/L), \quad \rho_0 = \rho_0(0)\left(1 - v_0^2/v_m^2\right), \quad B_0 = k_0\rho_0 \tag{4.3}$$
$$\xi = a_0\eta_\infty = Mc|B_0(0)|/4\pi em, \quad \eta = \eta_\infty \operatorname{th}^2(x/L)$$
$$v_m = |B_0(0)|\,[2\pi\rho_0(0)]^{-1/2}, \quad c_T^2 = \text{const} \ll c_A^2$$
$$f(x) = \frac{2f^*\operatorname{ch}^3(x/L)}{3\sqrt{3}\operatorname{sh}(x/L)}, \quad \eta_\infty = \frac{27}{4}\frac{L}{f^*}\left(\frac{v_m}{v_m f^*}\right)\left(\frac{c_T}{v_m}\right)^2,$$
$$\omega\tau = \frac{M\sigma|B_0|}{e\rho_0 c} = \text{const} \quad x \geqslant x_{00} > 0,$$
$$\rho_0(0) = \rho_0(x_{00})\left[1 - v_0^2(x_{00})/v_m^2\right]^{-1},$$
$$B_0(0) = k_0\rho_0(0).$$

Here $\omega\tau$ is the Hall parameter, ξ is the exchange parameter, and f^* is the width of the channel at the critical ($v_0 = c_A$) cross section. The true exchange parameter $\xi = I_d/I_m$ is determined by the expression $\xi = Mc|B_0(x_{00})|/4\pi em$, i.e., when $v_0(x_{00}) \ll v_m$ it differs little from the expression presented in (4.3). Since η_∞ does not depend on the Hall parameter, large values of ξ correspond to large values $a_0 = 2$. The assumption that $c_T^2 \ll c_A^2$ is observed when

$$0 \leqslant \eta/\eta_\infty < \eta_m/\eta_\infty = 1 - 2c_T^2/v_m^2. \tag{4.4}$$

Since we are assuming that the velocity at the channel entrance differs from zero, in Eqs. (4.3) one must set $x \geq x_{00} \geq 0$, i.e., $\eta \geq \eta_{00} > 0$, where values at the channel entrance are denoted by the subscript 00.

When $k^{(0)}(\eta, \psi) = 0$ the solution of Eq. (1.15) with the conditions (4.2) has the form

$$\begin{aligned} k_1 = {} & -k_0 \frac{P_1(\theta, \eta_{00})}{\rho_0(\eta_{00})\, c_T^2} + \exp\left(\frac{a_0}{2}\psi - \frac{a_0^2}{4}\eta\right) \int\limits_{\eta_{00}}^{\eta} d\mu \times \\ & \times \left[G(\eta-\mu, \psi) \frac{\partial}{\partial\mu} F(\theta, \mu) \exp\left(\frac{a_0^2}{4}\mu - \frac{a_0}{2}\right) + \right. \\ & \left. + G(\eta-\mu, 1-\psi) \frac{\partial}{\partial\mu} F(\theta,\mu) \exp\left(\frac{a_0^2}{4}\mu\right)\right] + \\ & + 2\sum_{n=1}^{\infty} \exp\left[(\eta_{00}-\eta)\left(\pi^2 n^2 + \frac{a_0^2}{4}\right)\right] \sin \pi n\psi \int\limits_0^1 \left[g(\theta, \xi) + \right. \\ & \left. + k_0 \frac{P_1(\theta, \eta_{00})}{\rho_0(\eta_{00})\, c_T^2}\right] \exp\left[\frac{a_0}{2}(\psi - \zeta)\right] \sin \pi n\zeta d\zeta. \end{aligned} \tag{4.5}$$

Here

$$G(\eta, \psi) = \psi + 2\sum_{n=1}^{\infty} (-1)^n \frac{\sin \pi n\psi}{\pi n} \exp(-\pi^2 n^2 \eta)$$

$$F(\theta, \mu) = \frac{k_0}{c_T^2}\left\{\frac{P_1(\theta, \eta_{00})}{\rho_0(\eta_{00})} - P_1\left(\theta + \int\limits_{\eta_{00}}^{\mu} d\nu/\Omega(\nu), \mu\right)[\rho_0(\mu)]^{-1}\right\}$$

and the functions $g(\theta, \psi)$ and $P_1(\theta, \eta)$ are reduced to zero when $\theta \leq 0$. It follows from (4.5) that the influence of the conditions at the channel entrance on the nature of the solution is important only when $0 < \eta - \eta_{00} \leqslant (\pi^2 + a_0^2/4)^{-1}$. At a given θ the function k_1 is essentially two-dimensional, i.e., it depends on η and ψ. One would think that if $[g(\theta, \psi) = -k_0 P_1(\theta, \eta_{00})/\rho_0(\eta_{00}) c_T^2$, then a solution $k_1 = -k_0 P_1(\theta, \eta_{00})/\rho_0 \cdot (\eta_{00}) c_T^2$ would exist, but in this case we would have $P_1 = P_1(\theta, \eta_{00})\, \rho_0(\eta)/\rho_0(\eta_{00})$, and such a dependence contradicts the integral equations (3.5).

The expression (4.5) for the function k_1 is complicated. Therefore, let us examine in more detail the ultra-Hall mode in which $a_0 \to \infty$ and $\xi \to \infty$. We will assume that $\eta - \eta_{00} \gg a_0^{-1} \gg a_0^{-2}$, so that the effect of the conditions at the channel entrance can be neglected. In this case k_1 has the form of (2.5):

$$g_1(\theta, \eta) + g_2(\theta, \eta) = -k_0 [\rho_0(\eta) c_T^2]^{-1} P_1\left(\theta + \int\limits_{\eta_{00}}^{\eta} d\mu/\Omega(\mu), \eta\right) = Q(\theta, \eta) \tag{4.6}$$

$$g_1(\theta, \eta - 1/a_0) + g_2(\theta, \eta + 1/a_0) \exp(a_0) = Q(\theta, \eta).$$

From this we have, approximately,

$$g_1(\theta, \eta) \approx Q(\theta, \eta) \tag{4.7}$$

$$g_2(\theta, \eta) \approx a_0^{-1} \exp(-a_0)\, \partial Q/\partial\eta.$$

In the variables (t, η, ψ) we have

$$k_1 = -\frac{k_0 P_1(t, \eta)}{\rho_0(\eta) c_T^2} + \frac{k_0 \exp[a_0(\psi - 1)]}{a_0 c_T^2}\left(\frac{\partial}{\partial\eta} + \frac{1}{\Omega(\eta)}\frac{\partial}{\partial t}\right)\frac{P_1(t, \eta)}{\rho_0(\eta)}. \tag{4.8}$$

Discarding the small terms of order a_0^{-2}, we obtain

$$\int\limits_0^1 k_1 d\psi = -k_0 P_1(t, \eta)/\rho_0 c_T^2, \quad \int\limits_0^1 \rho_1 d\psi = P_1(t, \eta)/c_T^2. \tag{4.9}$$

By introducing the function $W(t, x)$,

$$\int\limits_0^1 \frac{v_{1x}}{v_0} d\psi = \left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial x}\right) W, \quad \frac{P_1}{\rho_0 c_T^2} = -v_0 \frac{\partial W}{\partial x}, \tag{4.10}$$

we can satisfy the second integral equation (3.5). The first gives an equation for the function W:

$$\frac{\partial^2 W}{\partial t^2} + 2v_0 \frac{\partial^2 W}{\partial t \partial x} + \left(v_0^2 - c_T^2\right) \frac{\partial^2 W}{\partial x^2} + 2 \frac{\partial v_0}{\partial x} \left(\frac{\partial W}{\partial t} + v_0 \frac{\partial W}{\partial x}\right) - \frac{\partial W}{\partial x} \frac{c_T^2}{\rho_0 v_0} \frac{d}{\partial x} \rho_0 v_0 = 0. \tag{4.11}$$

Since the disturbance P_1 in the total pressure is caused by the given disturbance at the channel entrance, it must have the form of a departing wave. It is seen from (4.11) that for short-wavelength disturbances with a wavelength small in comparison with the scale of the variation in the parameters of the exponential flow the following relationship holds:

$$W = W\left(t - \int_{x_{00}}^{x} [v_0(\zeta) + c_T]^{-1} d\zeta\right). \tag{4.12}$$

The plus sign in front of c_T corresponds to a departing wave. The singularity arising upon the transition through the speed of sound c_T is fictitious and is connected with the approximation used. A similar singularity appears in the analysis of steady disturbances in the ultra-Hall approximation (see [1]).

Let us analyze the zone of transition through the speed of sound c_T. The latter is included in the region under consideration if $\xi \gg v_m^2/c_T^2$. Using a Fourier transformation with respect to time and expanding the parameters of the exponential flow near the transition point x_T by powers of $x - x_T$, we obtain the following equation for the Fourier component $s(\omega, x)$ of the function $W(t, x)$:

$$\lambda \frac{\partial^2 s}{\partial \lambda^2} + \alpha \frac{\partial s}{\partial \lambda} + \beta s = 0 \tag{4.13}$$

$$\lambda = (x - x_T)/L, \quad \alpha = (v_m'/L = 2i\omega) L (2v_m)^{-1}, \quad \beta = -(2iv_m/L + \omega) L^2 \omega (2c_T v_m)^{-1}.$$

A solution which is valid near the transition point is the function

$$s = s_0(\omega) \exp(-\beta\lambda/\alpha), \tag{4.14}$$

i.e.,

$$W = \int_{-\infty}^{\infty} s_0(\omega) \exp[i\omega t - \beta\lambda/\alpha] \, d\omega. \tag{4.15}$$

Far from the point of transition through the speed of sound, i.e., when $v_0 \gg c_T$ and $dv_0/dx \to 0$, the function W has the form (4.12), and one can neglect the value c_T in the denominator of the expression in the integral. From this analysis it follows that when a source of transient disturbances is turned on at the channel entrance the transient flow is established after a time interval $\tau \sim \tau_0 = \int_{x_{00}}^{x_m} d\zeta / v_0(\zeta)$, where x_m corresponds to the exit from the channel, and for an exponential flow one can set $x_m \to \infty$. When the source of transient disturbances is turned off the steady flow is established after the same time interval.

From (4.9) and (4.10) it follows that

$$\frac{\partial W}{\partial t} = \int_0^1 \left(\frac{\rho_1}{\rho_0} + \frac{v_{1x}}{v_0}\right) d\psi = \frac{\delta m^{\cdot}}{m^{\cdot}}, \tag{4.16}$$

where $\delta m^{\cdot}$ is the disturbance in the mass flow rate. Suppose that $\delta m^{\cdot}$ at the channel entrance varies sufficiently slowly – over times greater than the time of flight τ_0 of an ion through the channel. In this case, since the plasma does not precipitate onto the walls, one can consider the value of $\delta m^{\cdot}$ to be constant along the channel and neglect the second time derivative of the function W in Eq. (4.11). As a result we obtain

$$\frac{\delta m^{\cdot}}{m^{\cdot}} \frac{dv_0^2}{dx} + \frac{P_1}{\rho_0 c_T^2} \left(\frac{c_T^2}{\rho_0} \frac{d\rho_0}{\partial x} - v_0 \frac{dv_0}{dx}\right) = \left(v_0^2 - c_T^2\right) \frac{\partial}{\partial x} \frac{P_1}{\rho_0 c_T^2}. \tag{4.17}$$

Hence,

$$\frac{P_1}{\rho_0 c_T^2} = \frac{\delta m^{\cdot}}{m^{\cdot}} \int_{x_T}^{x} \frac{d\zeta}{v_0^2(\zeta) - c_T^2} \frac{dv_0^2}{d\zeta} \exp\left\{\int_{\zeta}^{x} \frac{d\mu \left(c_T^2 d\rho_0/d\mu - \rho_0 v_0 dv_0/d\mu\right)}{\rho_0(\mu) \left[v_0^2(\mu) - c_T^2\right]}\right. . \tag{4.18}$$

Substituting the parameters of the exponential flow, from (4.18) and (4.9) we obtain

$$\frac{P_1}{\rho_0 c_T^2} \approx \frac{2\delta m^{\cdot}}{m^{\cdot}}, \quad \int_0^1 \frac{\rho_1}{\rho_0} d\psi \approx \frac{2\delta m^{\cdot}}{m^{\cdot}}. \qquad \int_0^1 \frac{v_{1x}}{v_0} d\psi = -\frac{\delta m^{\cdot}}{m^{\cdot}}. \tag{4.19}$$

In the general case the value $\delta m^{\cdot}$ is described by the same equation (4.11) as the function W.

5. Let us consider the case in which the value of k_1 at the channel entrance does not depend on time, but the initial function $k^{(0)}(\eta, \psi)$ is different from zero (and does not satisfy the corresponding steady-state equation). The conditions on the function k_1 can be formulated in the form

$$k_1(t, \eta_{00}, \psi) = 0, \quad k_1(0, \eta, \psi) = k^{(0)}(\eta, \psi)$$
$$k_1(t, \eta, 0) = k_1(t, \eta, 1) = -k_0 P_1(t, \eta)/\rho_0 c_T^2. \tag{5.1}$$

In this case $P_1(t, \eta_{00}) = 0$ and $k^{(0)}(\eta, 0) = k^{(0)}(\eta, 1) = -k_0 P_1(0, \eta)/\rho_0 c_T^2$. If $k^{(0)} = 0$, then $P_1 = 0$ and $k_1(t, \eta, \psi) = 0$. The disturbances k_1 and P_1 are the result of the function $k^{(0)}(\eta, \psi)$ being different from zero. From (5.1) it follows that $k^{(0)}(\eta_{00}, \psi) = 0$. If after a certain time has elapsed the function k_1 ceases to depend on the concrete form of the function $k^{(0)}$, then this means that steady flow has been established.

The solution of Eq. (1.15) with the conditions (5.1) is found with the help of a Laplace transformation with respect to time, and for undisturbed exponential flow it has the form

$$k_1 = \exp\left(\frac{a_0}{2}\psi - \frac{a_0^2}{4}\eta\right)\int_{\eta_{00}}^{\eta} d\mu \left[G(\eta-\mu, \psi)\frac{\partial}{\partial\mu}F(\theta, \mu) \times \right.$$
$$\times \exp\left(\frac{a_0^2}{4}\mu - \frac{a_0}{2}\right) + G(\eta-\mu, 1-\psi)\frac{\partial}{\partial\mu}F(\theta, \mu)\exp\left(\frac{a_0^2}{4}\mu\right)\Bigg] +$$
$$+ \sum_{n=1}^{\infty}(-1)^n \exp\left[\eta(\varkappa-1)\left(\pi^2 n^2 + \frac{a_0^2}{4}\right) + \frac{a_0}{2}\psi\right] \times$$
$$\times \left[\int_0^{\psi} k^{(0)}(\varkappa\eta, \lambda)\exp\left(-\frac{a_0}{2}\lambda\right)\cos\pi n(\lambda + 1 - \psi)\, d\lambda + \right.$$
$$+ \int_{\psi}^{1} k^{(0)}(\varkappa\eta, \lambda)\exp\left(-\frac{a_0}{2}\lambda\right)\cos\pi n(\psi + 1 - \lambda)\, d\lambda -$$
$$\left. - \int_0^{1} k^{(0)}(\varkappa\eta, \lambda)\exp\left(-\frac{a_0}{2}\lambda\right)\cos\pi n(\psi - 1 + \lambda)\, d\lambda I\left[\ln\frac{\eta(\eta_\infty - \eta_{00})}{\eta_{00}(\eta_\infty - \eta)} - \frac{2t}{\tau_m}\right]\right], \tag{5.2}$$

$$\varkappa(t, \eta) = \frac{\eta_\infty \exp(-2t/\tau_m)}{\eta_\infty - \eta[1 - \exp(-2t/\tau_m)]}, \qquad \tau_m = \frac{L}{v_m}. \tag{5.3}$$

Here I is a unit function which is equal to zero for negative arguments. One can ascertain that the value of $\varkappa\eta$ depends only on θ.

The last term in Eq. (5.2) is interesting; it is reduced to zero when $\varkappa\eta = \eta_{00}$, i.e., when

$$t = t_0(\eta) = \frac{\tau_m}{2}\ln\frac{\eta(\eta_\infty - \eta_{00})}{\eta_{00}(\eta_\infty - \eta)}. \tag{5.4}$$

When $t > t_0(\eta)$ the solution at the given point η ceases to depend on the concrete form of the function $k^{(0)}(\eta, \psi)$, i.e., steady flow is established in the given cross section. The total time it takes to establish the flow in the channel is

$$\tau_0 = t_0(\eta_m) = \frac{\tau_m}{2}\ln\frac{v_m^4}{2c_T^2 v_0^2(x_{00})} \tag{5.5}$$

[we assume that $v_m^2 \gg c_T^2$ and $v_m^2 \gg v_0^2(x_{00})$]. It is seen that τ_0 is the time of flight of ions through the channel.

From the discussion presented in Secs. 4 and 5 it follows that the flow in the channel is established in a time on the order of the time of flight of the plasma (ions) through the channel (the region occupied by the magnetic field). The flow is stable. This is connected with the fact that the conductance of the plasma was assumed to be constant, and the condition that the vectors $\nabla\rho$ and ∇P are parallel is satisfied for the undisturbed flow.

LITERATURE CITED

1. A. I. Morozov and A. P. Shubin, "Flow of a plasma between electrodes having weak longitudinal conduction," Teplofiz. Vys. Temp., 3, No. 6 (1965).
2. G. M. Bam-Zelikovich, "Acceleration of a conducting gas in a strong transient electromagnetic field," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 6 (1968).

3. K. V. Brushlinskii, N. I. Gerlakh, and A. I. Morozov, "Effect of finite conductance on the two-dimensional flow of a plasma in a coaxial channel," Magnitn. Gidrodinam., No. 2 (1967).
4. S. A. Belyaev, G. V. Danilova, D. A. Gol'dina, L. V. Leskov, Yu. N. Kulikov, S. P. Kurdyumov, Yu. P. Popov, V. V. Savichev, A. A. Samarskii, S. S. Filippov, and L. S. Tsareva, "Calculation of the transient acceleration of a plasma in a one-dimensional approximation," Preprint In-ta Prikl. Matem. Akad. Nauk SSSR, No. 36 (1970).
5. K. V. Brushlinskii, N. I. Gerlakh, and A. I. Morozov, "Calculation of two-dimensional transient flows of a plasma of finite conductance in the presence of a Hall effect," Magnitn. Gidrodinam., No. 1 (1967).
6. A. I. Morozov and L. S. Solov'ev, "One similarity parameter in the theory of plasma flows," Dokl. Akad. Nauk SSSR, 164, No. 1 (1965).
7. K. V. Brushlinskii and A. I. Morozov, "Evolutionary nature of the equations of magnetohydrodynamics with allowance for the Hall effect," Prikl. Matem. i Mekhan., 32, No. 5 (1968).